JPRS: 4656

29 May 1961

ON THE THEORY OF OPTIMAL PROCESSES IN LINEAR SYSTEMS

By R. V. Gamkrelidze

-USSR-

Distributed by:

OFFICE OF TECHNICAL SERVICES
U. S. DEPARTMENT OF COMMERCE
WASHINGTON 25, D. C.

U. S. JOINT PUBLICATIONS RESEARCH SERVICE
1636 CONNECTICUT AVE., N.W.
WASHINGTON 25, D. C.

F O R E W O R D

This publication was prepared under contract by the UNITED STATES JOINT PUBLICATIONS RESEARCH SERVICE, a federal government organization established to service the translation and research needs of the various government departments.

JPRS: 4656

CSO: 1730-S/b

ON THE THEORY OF OPTIMAL PROCESSES IN LINEAR SYSTEMS *

-USSR-

[Following is the translation of an article by R.V. Gamkrelidze in Doklady Akademii Nauk SSSR (Reports of the USSR Academy of Sciences), Vol 116, No 1, 1957, pages 9-11]

(Presented by Academician P.S. Aleksandrov on 5 April 1957)

On the basis of the methods developed in reference (1), the present paper is concerned with the problem of finding optimal processes for systems with a single control parameter.

1. Statement of the problem. (see also (1)); notation. We are given a linear differential vector equation with one control (scalar) parameter u:

$$\dot{x} = Ax + \vec{b}u, \tag{1}$$

where x is a (vectorial) representative point in n-dimensional phase space X; $\vec{b}$ is a fixed vector in the same space, and A is a time-independent linear transform of space X. The control function u is chosen from a class of piecewise continuous functions (with a finite number of junction points) not exceeding 1 in absolute value: $|u| \leq 1$; we will term such functions "permissible".

We are given two points $\vec{\xi}_0, \vec{\xi}_1$ in phase space X; our aim is to select a permissible control u = u(t) such that the point x(t), moving along the locus of equation (1),

---

*The results presented in the present paper were obtained in L.S. Pontryagin's seminar on the mathematical problems of oscillation theory and automatic control.

will pass from point $\vec{\xi}_0$ to point $\vec{\xi}_1$ in the minimal time.

This we will an optimal control, and the corresponding locus -- an optimal locus.

Let us denote by $\vec{\varphi}_1(t),\dots,\vec{\varphi}_n(t)$ a sequence of contravariant functions having values in X, which forms the fundamental system of solutions of the equation $\dot{x} = Ax$. By $\vec{\psi}^1(t),\dots,\vec{\psi}^n(t)$ we denote the covariant vector functions in one-to-one correspondence to the functions $\vec{\varphi}_i(t)$: $\vec{\varphi}_i(t)\cdot\psi^j(t)=\delta_i^j$. We then have:

$$\dot{\vec{\varphi}}_i(t) = A\vec{\varphi}_i(t), \quad \dot{\vec{\psi}}^i(t) = -A'\vec{\psi}^i(t), \qquad (2)$$

where A' is a linear transform conjugate to A. Let us introduce n functions $h^i(t) = \vec{\psi}^i(t)\cdot b$, $i = 1,\dots,n$. The solution x(t) of equation (1) with the initial condition $x(0) = \vec{\xi} = \varphi_\alpha(0)\xi^\alpha$ is written in the form

$$x(t) = \vec{\varphi}_\alpha(t)\left(\xi^\alpha + \int_0^t h^\alpha(\tau)u(\tau)\,d\tau\right).$$

2. Condition for non-degeneracy. Let us call equation (1) non-degenerate if vector $\vec{b}$ does not lie in any invariant subspace of dimensionality $\leq n-1$ of transform A.

If equation (1) is degenerate, then either the time of passage from $\vec{\xi}_0$ to $\vec{\xi}_1$ is independent of the choice of control function u, or the problem is reduced to the analogous one for lower-order equations.

Henceforth, it is assumed that equation (1) is non-degenerate. In this case vectors $\vec{b}$, $A\vec{b},\dots,A^{n-1}\vec{b}$ are independent, so that the functions $h^1(t),\dots,h^n(t)$ are linearly independent.

3. Optimal control and optimal locus equations. Let u(t) be an optimal control, and x(t) the corresponding optimal locus connecting points $\vec{\xi}_0$ and $\vec{\xi}_1$; $x(0) = \vec{\xi}_0$, $x(t_1) = \vec{\xi}_1$. Then through every point of the locus x(t), $0 \leq t \leq t_1$ it will be possible to draw an (n-1)-dimensional hyperplane which satisfies the following condition.

Let us denote by $\vec{\psi}(t)$ a covariant vector which is orthogonal to the hyperplane drawn through point x(t) of the locus and uniquely defines this hyperplane. It turns out that for any permissible control $u(t) + \delta u(t)$ and the corresponding locus $x(t) + \delta x(t)$, where $\delta x(0)=0$, we have satisfied the condition

$$\vec{\psi}(t)\cdot\delta x(t) \leq 0,$$

while the vector function $\vec{\psi}(t)$ can be chosen so that it will satisfy the following differential equation

$$\dot{\vec{\psi}} = -A'\vec{\psi}. \qquad (3)$$

Consequently, $\vec{\psi}(t) = c_\alpha \vec{\psi}^\alpha(t)$, and we have:

$$\vec{\psi}(t)\cdot\delta x(t) = c_\alpha \vec{\psi}^\alpha(t)\cdot\vec{\varphi}_\beta(t)\int_\alpha^t h^\beta \delta u \, d\tau = \int_0^t \vec{\psi}\cdot\vec{b}\,\delta u \, d\tau \leq 0$$

Since the functions $h^\alpha(t)$, $\alpha = 1,\dots,n$ are linearly independent, $\vec{\psi}(t)\cdot\vec{b}$ is a nonzero solution of the linear n^{th}-order differential equation, and from the fact that $\delta u(t)$ is an arbitrary permissible variation of the optimal equation u(t), there follows the equation

$$u(t) = \operatorname{sign} \vec{\psi}(t)\cdot\vec{b} \qquad (4)$$

We have yet another condition which is satisfied:

$$\vec{\psi}(t)\cdot\dot{x}(t) = \vec{\psi}(t)\cdot[Ax(t) + \vec{b}u(t)] = \text{const} \geq 0. \qquad (5)$$

Combining the equations (1), (3) - (5), we obtain the theorem:

All of the optimal controls u(t) and the corresponding optimal loci x(t) which start out from the point ξ at t = 0 are contained in the controls and corresponding loci obtained from the solution of the following system of equations:

$$\dot{x} = Ax + \vec{b}u, \; x(0) = \vec{\xi}; \; \dot{\vec{\psi}} = -A'\vec{\psi}, \; u = \operatorname{sign}\vec{\psi}\cdot\vec{b} \qquad (6)$$

The initial value $\vec{\psi}(0)$ of the solution $\vec{\psi}(t)$ is subject to a single condition:

$$\vec{\psi}(0)\cdot[Ax(0) + \vec{b}u(0)] \geq 0.$$

The system of equations (6) precisely expresses the Maximum Principle formulated in reference (1).

4. <u>The problem of optimal systems synthesis.</u> The theory of automatic control is concerned with optimal passage along the locus of equation (1) from an arbitrary initial point x to the origin of the coordinate system. The set M of points x in phase space X from which it is possible to reach the origin by means of a permissible, and consequently, an optimal control, is an open convex set. If the transform A has stable proper values, the set M

corresponds to the entire space X.

There is not more than one locus of equation (1) from an arbitrary point x to the origin of the coordinate system which satisfies the Maximum Principle (6).

Consequently, we have a uniquely-determined real function u(x) of the vectorial argument x which has the property that in moving along the locus of the equation

$$\dot{x} = Ax + \vec{b}u(x),$$

we will reach the origin of coordinates from any given initial position in the minimum length of time, provided that the origin can be reached from this point by means of any permissible control.

The calculation of a function u(x) is called the synthesis of an optimal system (1). This calculation can be performed on the basis of formulas (6). Having set an arbitrary value $\vec{\psi}(0)$ for $\vec{\psi}$ and the initial value x(0) = 0, it is necessary to solve the system (6) along the semi-axis $-\infty < t \leq 0$. Since the function u(x) takes on the three values 1,0,-1, it is sufficient for its determination to know the set of "transformation" points of the control u(x), i.e., the set of values of x which satisfy the equation u(x), as well the regions into which this set divides the space X.

In case of a second-order equation, transformation lines are forms in the phase plane whose determination on the basis of equation (6) constitutes a perfectly elementary problem. These lines were first obtained by Bushow (see ref. (2)).

If the transform A takes on real proper values, then the set of transformation points of the control function u(x) represents a hypersurface. The method of constructing this hypersurface was first indicated by A.A. Fel'dbaum (ref. 3). On the basis of equations (6) it is easy to obtain a parametric representation of this hypersurface. Let $t_1, t_2, \ldots, t_{n-1}$ be parameters subject to the single condition $0 \geq t_1 \geq t_2 \geq t_3 \geq \cdots \geq t_{n-1}$; then the parametric representation of the transformation hyperspace will be

$$x(t_1, \ldots, t_{n-1}) = \pm \vec{\varphi}_\alpha(t_{n-1}) \left[\int_0^{t_1} \psi^\alpha \cdot \vec{b}\, d\tau - \int_{t_1}^{t_2} \vec{\psi}^\alpha \cdot \vec{b}\, d\tau + \cdots + (-1)^{n-2} \int_{t_{n-2}}^{t_{n-1}} \vec{\psi}^\alpha \cdot \vec{b}\, d\tau \right]. \quad (7)$$

This hypersurface divides the space into two adjoint regions; in one of these the function takes on the value +1, and a

value of -1 in the other.

In the general case involving complex roots, the set of transformation points of the control function u is a pseudomanifold, and its parametric representation, which is analogous to (7), is impossible to obtain. It can, however, be calculated on the basis of equations (6) to any degree of accuracy.

Mathematics Institute imeni V.A. Steklov
of the USSR Academy
of Sciences

Received
4 April 1957

References

1. V.G. Boltyanskiy, R.V. Gamkrelidze, L.S. Pontryagin, Doklady Akademii Nauk SSSR (Reports of the USSR Academy of Sciences), 110, No 1, 7(1956).
2. Tsian Hsu-hsien (Tsyan' Syue-sen'), Technological Cybernetics, Chapter X, Foreign Languages Publishing House, 1955.
3. A.A. Fel'dbaum, Avtomatika i Telemekhanika (Automation and Remote Control), 16, No 2, 129(1955).

/10,443-Ed.

-END-